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# FOURIER-JACOBI TYPE SPHERICAL FUNCTIONS ON $Sp(2, \mathbf{R})$ ; THE CASE OF $P_J$ -PRINCIPAL SERIES AND DISCRETE SERIES

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## 1. Introduction

In this note, we study a kind of generalized Whittaker models, or equally, of generalized spherical functions associated with automorphic forms on the real symplectic group of degree two. We call these spherical functions 'Fourier-Jacobi type', since these are closely connected with the coefficients of the 'Fourier-Jacobi expansions' of (holomorphic or non-holomorphic) automorphic forms. Also these can be considered as a non-holomorphic analogue of the local Whittaker-Shintani functions on  $Sp(2, \mathbf{R})$  of Fourier-Jacobi type in the paper of Murase and Sugano [6].

## 2. Preliminaries

*2.1. Groups and algebras.* We denote by  $\mathbf{Z}_{\geq m}$  the set of integers  $n$  such that  $n \geq m$ . Moreover, we use the convention that unwritten components of a matrix are zero.

Let  $G$  be the real symplectic group  $Sp(2, \mathbf{R})$  of degree two given by

$$Sp(2, \mathbf{R}) = \left\{ g \in M_4(\mathbf{R}) \mid {}^t g J_2 g = J_2 = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}, \det g = 1 \right\}.$$

Let  $\theta(g) = {}^t \bar{g}^{-1}$  ( $g \in G$ ) be a Cartan involution of  $G$  and  $K$  be the set of fixed points of  $\theta$ . Then  $K$  becomes a maximal compact subgroup of  $G$  which is isomorphic to the unitary group  $U(2)$ .

Let  $\mathfrak{g} = \{X \in M_4(\mathbf{R}) \mid J_2 X + {}^t X J_2 = 0\}$  be the Lie algebra of  $G$ . If we denote the differential of  $\theta$  again by  $\theta$ , then we have  $\theta(X) = -{}^t \bar{X}$  ( $X \in \mathfrak{g}$ ). Let  $\mathfrak{k}$  and  $\mathfrak{p}$  be the  $+1$  and  $-1$  eigenspaces of  $\theta$  in  $\mathfrak{g}$ , respectively, and hence

$$\begin{aligned} \mathfrak{k} &= \left\{ X \in \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid A, B \in M_2(\mathbf{R}), {}^t A = -A, {}^t B = B \right\}, \\ \mathfrak{p} &= \left\{ X \in \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \mid A, B \in M_2(\mathbf{R}), {}^t A = A, {}^t B = B \right\}. \end{aligned}$$

Then we have a Cartan decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ . Of course,  $\mathfrak{k}$  is the Lie algebra of  $K$  which is isomorphic to the unitary algebra  $\mathfrak{u}(2)$ .

For a Lie algebra  $\mathfrak{l}$ , we denote by  $\mathfrak{l}_{\mathbb{C}} = \mathfrak{l} \otimes_{\mathbb{R}} \mathbb{C}$  the complexification of  $\mathfrak{l}$ . Let  $\mathfrak{h}$  be a compact Cartan subalgebra of  $\mathfrak{g}$  given by

$$\mathfrak{h} = \left\{ H(\theta_1, \theta_2) = \left( \begin{array}{c|c} & \theta_1 \\ \hline & \theta_2 \\ -\theta_1 & \\ \hline & -\theta_2 \end{array} \right) \middle| \theta_i \in \mathbb{R} \right\}.$$

Now we identify a linear form  $\beta : \mathfrak{h}_{\mathbb{C}} \rightarrow \mathbb{C}$  with  $(\beta_1, \beta_2) \in \mathbb{C}^2$  via  $\beta = \beta_1 e_1 + \beta_2 e_2$ , where  $e_i(H(\theta_1, \theta_2)) = \sqrt{-1}\theta_i$ . Then the set of roots  $\Delta = \Delta(\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$  of  $(\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$  is given by

$$\Delta = \{\pm(2, 0), \pm(0, 2), \pm(1, 1), \pm(1, -1)\}.$$

Fix a positive root system  $\Delta^+ = \{(2, 0), (0, 2), (1, 1), (1, -1)\}$ , and put  $\Delta_c^+$  and  $\Delta_n^+$  the set of compact and non-compact positive roots, respectively. Then

$$\Delta_c^+ = \{(1, -1)\}, \quad \Delta_n^+ = \{(2, 0), (0, 2), (1, 1)\}.$$

If we denote the root space for  $\beta \in \Delta$  by  $\mathfrak{g}_{\beta}$ , then we have a decomposition  $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$  with  $\mathfrak{p}_+ = \sum_{\beta \in \Delta_n^+} \mathfrak{g}_{\beta}$  and  $\mathfrak{p}_- = \sum_{\beta \in \Delta_n^+} \mathfrak{g}_{-\beta}$ .

Put  $P_J$  the Jacobi maximal parabolic subgroup of  $G$  with the Langlands decomposition  $P_J = M_J A_J N_J$ , where

$$M_J = \left\{ \left( \begin{array}{c|c} \varepsilon & \\ \hline a & b \\ \hline c & \varepsilon \\ \hline & d \end{array} \right) \middle| \varepsilon \in \{\pm 1\}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) \right\} \simeq \{\pm I\} \times SL(2, \mathbb{R}),$$

$$N_J = \left\{ n(x, y, z) = \left( \begin{array}{cc|c} 1 & y & \\ & 1 & \\ \hline & & 1 \\ & & -y & 1 \end{array} \right) \cdot \left( \begin{array}{cc|c} 1 & & z & x \\ & 1 & x & \\ \hline & & 1 & \\ & & & 1 \end{array} \right) \middle| x, y, z \in \mathbb{R} \right\},$$

and  $A_J = \{\text{diag}(a, 1, a^{-1}, 1) \mid a > 0\}$ . Remark that the unipotent radical  $N_J$  of  $P_J$  is isomorphic to the 3-dimensional Heisenberg group  $\mathcal{H}_1$ . The Levi part  $M_J A_J$  of  $P_J$  acts on  $N_J$  via the conjugate action, and  $M_J$  gives the centralizer of the center  $Z(N_J) = \{n(0, 0; z) \mid z \in \mathbb{R}\} \simeq \mathbb{R}$  of  $N_J$  in  $M_J A_J$ . Now we define the Jacobi group  $R_J$  by the semidirect product  $M_J^{\circ} \ltimes N_J \simeq SL(2, \mathbb{R}) \ltimes \mathcal{H}_1$ , where  $M_J^{\circ} \simeq SL(2, \mathbb{R})$  is the identity component of  $M_J$ .

**2.2. Representations.** First we investigate the irreducible unitary representations of the Jacobi group  $R_J$ . Since  $Z(R_J) = Z(N_J) \simeq \mathbb{R}$ , the central characters of elements in  $\hat{R}_J$  and  $\hat{N}_J$  are parametrized by the real numbers. Then we call an irreducible unitary representation of  $R_J$  and  $N_J$  of type  $m$  if its central character is of the form  $z \mapsto e^{2\pi\sqrt{-1}mz}$  with  $m \in \mathbb{R}$ . Let  $\nu \in \hat{N}_J$  of type  $m$ . According to the

theorem of Stone-von Neumann (cf. Corwin-Greenleaf [1; pp.46-47, 51-52]),  $\nu$  is a character if  $m = 0$  and  $\nu$  is infinite dimensional if  $m \neq 0$ . Moreover  $\nu$  of type  $m \neq 0$  is uniquely determined by  $m$  up to unitary equivalence. Now we fix an irreducible unitary representation  $(\nu_m, \mathcal{U}_m)$  of  $N_J$  of type  $m \neq 0$ . From the theory of the Weil representation,  $(\nu_m, \mathcal{U}_m)$  can be extended to a continuous true projective unitary representation  $(\tilde{\nu}_m, \mathcal{U}_m)$  of  $R_J$  by  $\tilde{\nu}_m(\tilde{n}) = W_m(g)\nu_m(n)$  for  $\tilde{n} = g \cdot n \in M_J^\circ \ltimes N_J$  with the Weil representation  $W_m$  on  $M_J^\circ$ . Here  $\tilde{\nu}_m$  has a factor set  $\alpha$  which is a proper 2-cocycle.

**Lemma 2.1.** (Satake [7; Appendix I, Proposition 2]) *Let  $\tilde{\nu}_m$  ( $m \neq 0$ ) as above. For every irreducible projective unitary representation  $\pi$  of  $M_J^\circ$  with factor set  $\alpha^{-1}$ , put  $\rho(\tilde{n}) = \pi(g) \otimes \tilde{\nu}_m(\tilde{n})$  for  $\tilde{n} = g \cdot n \in M_J^\circ \ltimes N_J$ . Then  $\rho$  is an irreducible unitary representation of  $R_J$ . Conversely, all irreducible unitary representations of  $R_J$  of type  $m \neq 0$  are obtained in this manner. Moreover  $\rho$  is square-integrable iff  $\pi$  is so.*

Let  $(\rho, \mathcal{F}_\rho)$  be an irreducible unitary representation of  $R_J$  of type  $m \neq 0$ . From the above lemma, we can regard  $(\rho, \mathcal{F}_\rho) \in \hat{R}_J$  as a tensor product representation  $(\pi_1 \otimes \tilde{\nu}_m, \mathcal{W}_{\pi_1} \otimes \mathcal{U}_m)$ . Here, if we write  $\widetilde{M}_J^\circ$  for the double cover of  $M_J^\circ \simeq SL(2, \mathbf{R})$ ,  $(\tilde{\nu}_m, \mathcal{U}_m)$  is a unitary representation of  $\widetilde{M}_J^\circ \ltimes N_J$  which is extended from  $(\nu_m, \mathcal{U}_m) \in \hat{N}_J$  as above and  $(\pi_1, \mathcal{W}_{\pi_1})$  is a unitary representation of  $\widetilde{M}_J^\circ$  which does not factor through  $M_J^\circ$ . On the other hand, the unitary dual of  $\widetilde{M}_J^\circ$  is given as follows.

**Proposition 2.2.** (cf. Gelbert[2; Lemma 4.1, 4.2]) *The following representations exhaust a set of representatives for the equivalence classes of irreducible unitary representations of  $\widetilde{SL}(2, \mathbf{R})$ .*

- (1) (unitary principal series)  $\mathcal{P}_s^\tau$ ,  $s \in \sqrt{-1}\mathbf{R}$ ,  $\tau = 0, 1, \pm\frac{1}{2}$  except for the case  $(s, \tau) = (0, 1)$ .
- (2) (complementary series)  $\mathcal{C}_s^\tau$ ,  $0 < s < 1$  for  $\tau = 0, 1$  and  $0 < s < \frac{1}{2}$  for  $\tau = \pm\frac{1}{2}$ .
- (3) ((limit of) discrete series)  $\mathcal{D}_k^\pm$ ,  $k \in \frac{1}{2}\mathbf{Z}_{\geq 2}$ .
- (4) (quotient representation)  $\mathcal{D}_{\frac{1}{2}}^-, \mathcal{D}_{\frac{1}{2}}^+$ .
- (5) The trivial representation of  $SL(2, \mathbf{R})$ .

In the above, the representations  $\mathcal{P}_s^\tau$ ,  $\mathcal{C}_s^\tau$  for  $\tau = 0, 1$ ,  $\mathcal{D}_k^\pm$  for  $k \in \mathbf{Z}_{\geq 1}$  and (5) factor through  $SL(2, \mathbf{R})$ , and the otherwise not.

Hence we take as  $(\pi_1, \mathcal{W}_{\pi_1})$  one of the irreducible unitary representations  $\mathcal{P}_s^\tau, \mathcal{C}_s^\tau$  with  $\tau = \pm\frac{1}{2}$  and  $\mathcal{D}_k^\pm$  with  $k \in \frac{1}{2}\mathbf{Z} \setminus \mathbf{Z}$ ,  $k \geq \frac{1}{2}$ .

**Remark 2.3.** The Weil representation  $W_m$  considered as the representation of  $\widetilde{M}_J^\circ$  has the following irreducible decomposition;

$$W_m = \begin{cases} \mathcal{D}_{\frac{1}{2}}^+ \oplus \mathcal{D}_{\frac{3}{2}}^+, & \text{if } m > 0, \\ \mathcal{D}_{\frac{1}{2}}^- \oplus \mathcal{D}_{\frac{3}{2}}^-, & \text{if } m < 0. \end{cases}$$

Next, we treat the irreducible unitary representations of  $K$ . Since  $\Delta_c^+$  is also a positive system of  $\Delta(\mathfrak{k}_\mathbf{C}, \mathfrak{h}_\mathbf{C})$ , then the set of the  $\Delta_c^+$ -dominant weights, and thus

$\hat{K}$ , is parametrized by the set

$$\Lambda = \{\lambda = (\lambda_1, \lambda_2) \mid \lambda_i \in \mathbf{Z}, \lambda_1 \geq \lambda_2\}$$

(cf. Knapp[4; Theorem 4.28]). We denote by  $(\tau_\lambda, V_\lambda)$  the element of  $\hat{K}$  corresponding to  $\lambda = (\lambda_1, \lambda_2) \in \Lambda$ . Here  $\dim V_\lambda = d_\lambda + 1$  with  $d_\lambda = \lambda_1 - \lambda_2$ .

Both of  $\mathfrak{p}_\pm$  become  $K$ -modules via the adjoint representation of  $K$ , and we have isomorphisms  $\mathfrak{p}_+ \simeq V_{(2,0)}$  and  $\mathfrak{p}_- \simeq V_{(0,-2)}$ . For a given irreducible  $K$ -module  $V_\lambda$  with the parameter  $\lambda = (\lambda_1, \lambda_2) \in \Lambda$ , the tensor product  $K$ -modules  $V_\lambda \otimes \mathfrak{p}_+$  and  $V_\lambda \otimes \mathfrak{p}_-$  have the irreducible decompositions

$$V_\lambda \otimes \mathfrak{p}_+ \simeq \bigoplus_{\beta \in \Delta_n^+} V_{\lambda+\beta}, \quad V_\lambda \otimes \mathfrak{p}_- \simeq \bigoplus_{\beta \in \Delta_n^+} V_{\lambda-\beta}.$$

For each  $\beta \in \Delta_n^+$ , let  $P^\beta : V_\lambda \otimes \mathfrak{p}_+ \rightarrow V_{\lambda+\beta}$  and  $P^{-\beta} : V_\lambda \otimes \mathfrak{p}_- \rightarrow V_{\lambda-\beta}$  be the projectors into the irreducible factors of  $V_\lambda \otimes \mathfrak{p}_\pm$ .

In this note, we consider the following two series of representations of  $G$ ; one is the principal series induced from  $P_J$ , and the other is the discrete series. We explain these representations in the remaining of this section.

Let  $\sigma = (\varepsilon, D)$  be a representation of  $M_J \simeq \{\pm I\} \times SL(2, \mathbf{R})$  with a character  $\varepsilon : \{\pm I\} \rightarrow \mathbf{C}^\times$  and a discrete series representation  $D = \mathcal{D}_n^\pm$  ( $n \in \mathbf{Z}_{\geq 2}$ ) of  $SL(2, \mathbf{R})$ , and take a quasi-character  $\nu_z$  ( $z \in \mathbf{C}$ ) of  $A_J$  such that  $\nu_z(\text{diag}(a, 1, a^{-1}, 1)) = a^z$ . Then we can construct a induced representation  $\text{Ind}_{P_J}^G(\sigma \otimes \nu_z \otimes 1_{N_J})$  of  $G$  from the Jacobi maximal parabolic subgroup  $P_J = M_J A_J N_J$  by the usual manner (cf. Knapp[4; Chapter VII]), and call  $\text{Ind}_{P_J}^G(\sigma \otimes \nu_z \otimes 1_{N_J})$  the  $P_J$ -principal series representation of  $G$ . The following lemma is derived from the Frobenius reciprocity for induced representations.

**Lemma 2.4.**  $\tau_\lambda \in \hat{K}$  with the parameter  $\lambda = (\lambda_1, \lambda_2) \in \Lambda$  such that  $\lambda_1 < n$  (resp.  $\lambda_2 > -n$ ) does not occur in the  $K$ -type of  $\text{Ind}_{P_J}^G(\sigma \otimes \nu_z \otimes 1_{N_J})$  for  $D = \mathcal{D}_n^+$  (resp.  $\mathcal{D}_n^-$ ). The 'corner'  $K$ -types  $\tau_\lambda \in \hat{K}$  of  $\text{Ind}_{P_J}^G(\sigma \otimes \nu_z \otimes 1_{N_J})$  with the parameter  $\lambda \in \Lambda$  given below occur with multiplicity one.

- (1)  $\lambda = (n, n)$  for  $\varepsilon(\gamma) = (-1)^n$  and  $D = \mathcal{D}_n^+$ ,
- (2)  $\lambda = (n, n-1)$  for  $\varepsilon(\gamma) = -(-1)^n$  and  $D = \mathcal{D}_n^+$ ,
- (3)  $\lambda = (-n, -n)$  for  $\varepsilon(\gamma) = (-1)^n$  and  $D = \mathcal{D}_n^-$ ,
- (4)  $\lambda = (-n+1, -n)$  for  $\varepsilon(\gamma) = -(-1)^n$  and  $D = \mathcal{D}_n^-$ .

Here  $\gamma = \text{diag}(-1, 1, -1, 1)$ .

In order to parametrize the discrete series representations of  $G$ , we enumerate all the positive root systems compatible to  $\Delta_c^+$ :

- (I)  $\Delta_I^+ = \{(1, -1), (2, 0), (1, 1), (0, 2)\}$ ,
- (II)  $\Delta_{II}^+ = \{(1, -1), (2, 0), (1, 1), (0, -2)\}$ ,
- (III)  $\Delta_{III}^+ = \{(1, -1), (2, 0), (0, -2), (-1, -1)\}$ ,
- (IV)  $\Delta_{IV}^+ = \{(1, -1), (0, -2), (-1, -1), (-2, 0)\}$ .

Let  $J$  be a variable running over the set of indices I, II, III, IV, and let us denote the set of non-compact positive roots for the index  $J$  by  $\Delta_{J,n}^+ = \Delta_J^+ - \Delta_c^+$ . Define a subset  $\Xi_J$  of  $\Delta_c^+$ -dominant weights by

$$\Xi_J = \{ \Lambda = (\Lambda_1, \Lambda_2), \Delta_c^+ - \text{dominant weight} \mid \langle \Lambda, \beta \rangle > 0, \forall \beta \in \Delta_{J,n}^+ \}.$$

The set  $\bigcup_{J=I}^{IV} \Xi_J$  gives the Harish-Chandra parametrization of the discrete series representation of  $G$ . Let us write by  $\pi_\Lambda$  the discrete series representation of  $G$  with the Harish-Chandra parameter  $\Lambda \in \bigcup_{J=I}^{IV} \Xi_J$ . Then  $\pi_\Lambda$  is called *the holomorphic* discrete series representation if  $\Lambda \in \Xi_I$  and *the anti-holomorphic* one if  $\Lambda \in \Xi_{IV}$ . Moreover if  $\Lambda \in \Xi_{II} \cup \Xi_{III}$ , a discrete series representation  $\pi_\Lambda$  is called *large* (in the sense of Vogan[8]).

The Blattner formula gives the description of the  $K$ -types of  $\pi_\Lambda$ . In particular, the minimal  $K$ -type  $(\tau_\lambda, V_\lambda)$  of  $\pi_\Lambda$  is given by the formula  $\lambda = \Lambda - \rho_c + \rho_n$ , where  $\rho_c$  (resp.  $\rho_n$ ) is the half sum of compact (resp. non-compact) positive roots in  $\Delta_J^+$ . We call such  $\lambda$  *the Blattner parameter* of  $\pi_\Lambda$ .

### 3. Fourier-Jacobi type spherical functions

**3.1. Radial parts.** Let  $(\rho, \mathcal{F}_\rho)$  be an irreducible unitary representation of  $R_J$  and let  $(\tau, V_\tau)$  be a finite dimensional  $K$ -module. We denote by  $C_{\rho,\tau}^\infty(R_J \backslash G/K)$  the space of smooth functions  $F : G \rightarrow \mathcal{F}_\rho \otimes V_\tau$  with the property

$$F(r g k) = (\rho(r) \otimes \tau(k)^{-1}) F(g), \quad (r, g, k) \in R_J \times G \times K.$$

On the other hand, let  $C^\infty(A_J; \rho, \tau)$  be the space of smooth functions  $\varphi : A_J \rightarrow \mathcal{F}_\rho \otimes V_\tau$  satisfying

$$(\rho(m) \otimes \tau(m)) \varphi(a) = \varphi(a), \quad m \in R_J \cap K = M_J^\circ \cap K, \quad a \in A_J.$$

Because of an Iwasawa decomposition of  $G$ , we have  $G = R_J A_J K$ . Also we remark that all elements in  $M_J^\circ \cap K$  are commutative with  $a \in A_J$ . Then the restriction to  $A_J$  gives a linear map from  $C_{\rho,\tau}^\infty(R_J \backslash G/K)$  to  $C^\infty(A_J; \rho, \tau)$ , which is injective. For each  $f \in C_{\rho,\tau}^\infty(R_J \backslash G/K)$ , we call  $f|_{A_J} \in C^\infty(A_J; \rho, \tau)$  *the radial part* of  $f$ , where  $|_{A_J}$  means the restriction to  $A_J$ .

Let  $(\tau', V_{\tau'})$  be also a finite dimensional  $K$ -module. For each  $\mathbf{C}$ -linear map  $u : C_{\rho,\tau}^\infty(R_J \backslash G/K) \rightarrow C_{\rho,\tau'}^\infty(R_J \backslash G/K)$ , we have a unique  $\mathbf{C}$ -linear map  $\mathcal{R}(u) : C^\infty(A_J; \rho, \tau) \rightarrow C^\infty(A_J; \rho, \tau')$  with the property  $(uf)|_{A_J} = \mathcal{R}(u)(f|_{A_J})$  for  $f \in C_{\rho,\tau}^\infty(R_J \backslash G/K)$ . We call  $\mathcal{R}(u)$  *the radial part* of  $u$ .

**3.2. Fourier-Jacobi type spherical functions.** Let  $(\rho, \mathcal{F}_\rho)$  be as above and consider a  $C^\infty$ -induced representation  $C^\infty \text{Ind}_{R_J}^G(\rho)$  with the representation space

$$C_\rho^\infty(R_J \backslash G) = \{ F : G \rightarrow \mathcal{F}_\rho, \quad C^\infty \mid F(r g) = \rho(r) F(g), \quad (r, g) \in R_J \times G \}$$

on which  $G$  acts by the right translation. Then  $C_\rho^\infty(R_J \backslash G)$  becomes a smooth  $G$ -module and a  $(\mathfrak{g}_\mathbf{C}, K)$ -module naturally. Moreover let  $(\tau, V_\tau) \in \hat{K}$  and take an

irreducible Harish-Chandra module  $\pi$  of  $G$  with the  $K$ -type  $\tau^*$ , where  $\tau^*$  is the contragredient representation of  $\tau$ . Now we consider the intertwining space

$$\mathcal{I}_{\rho,\pi} := \text{Hom}_{(\mathfrak{g}_{\mathbb{C}}, K)}(\pi, C^\infty \text{Ind}_{R_J}^G(\rho))$$

between  $(\mathfrak{g}_{\mathbb{C}}, K)$ -modules and its restriction to the  $K$ -type  $\tau^*$  of  $\pi$ .

Let  $i : \tau^* \rightarrow \pi|_K$  be a  $K$ -equivariant map and let  $i^*$  be the pullback via  $i$ . Then the map

$$\mathcal{I}_{\rho,\pi} \xrightarrow{i^*} \text{Hom}_K(\tau^*, C_{\rho}^\infty(R_J \backslash G)) \simeq C_{\rho,\tau}^\infty(R_J \backslash G/K)$$

gives the restriction of  $T \in \mathcal{I}_{\rho,\pi}$  to the  $K$ -type  $\tau^*$  and we denote the image of  $T$  in  $C_{\rho,\tau}^\infty(R_J \backslash G/K)$  by  $T_i$ . Now the space  $\mathcal{J}_{\rho,\pi}(\tau)$  of the algebraic Fourier-Jacobi type spherical functions of type  $(\rho, \pi; \tau)$  on  $G$  is defined by

$$\mathcal{J}_{\rho,\pi}(\tau) := \bigcup_{i \in \text{Hom}_K(\tau^*, \pi|_K)} \{T_i \mid T \in \mathcal{I}_{\rho,\pi}\}.$$

Moreover put

$$\mathcal{J}_{\rho,\pi}^\circ(\tau) = \{f \in \mathcal{J}_{\rho,\pi}(\tau) \mid f|_{A_J}(\text{diag}(a, 1, a^{-1}, 1)) \text{ is of moderate growth as } a \rightarrow \infty\}.$$

We call  $f \in \mathcal{J}_{\rho,\pi}^\circ(\tau)$  a Fourier-Jacobi type spherical functions of type  $(\rho, \pi; \tau)$ .

In this note, we investigate the space  $\mathcal{J}_{\rho,\pi}^\circ(\tau)$  for the following triplet  $(\rho, \pi; \tau)$ : As  $\pi \in \hat{G}$  and  $\tau^* \in \hat{K}$ , we take either the  $P_J$ -principal series representation and the corner  $K$ -type or the discrete series representation and the minimal  $K$ -type, and also as  $\rho \in \hat{R}_J$  the one with the non-trivial central character, i.e. of type  $m \neq 0$ .

#### 4. Differential equations

*4.1. Differential operators.* In this subsection, we introduce some differential operators acting on  $C_{\rho,\tau}^\infty(R_J \backslash G/K)$ .

Take an orthonormal basis  $\{X_i\}$  of  $\mathfrak{p}$  with respect to the Killing form of  $\mathfrak{g}$ . Now we define a first order gradient type differential operator

$$\nabla_{\rho,\tau} : C_{\rho,\tau}^\infty(R_J \backslash G/K) \rightarrow C_{\rho,\tau \otimes \text{Ad}_{\mathfrak{p}_{\mathbb{C}}}}^\infty(R_J \backslash G/K)$$

by

$$\nabla_{\rho,\tau} f = \sum_i R_{X_i} f \otimes X_i, \quad f \in C_{\rho,\tau}^\infty(R_J \backslash G/K),$$

where

$$R_X f(g) = \left. \frac{d}{dt} f(g \cdot \exp(tX)) \right|_{t=0}, \quad X \in \mathfrak{g}_{\mathbb{C}}, \quad g \in G.$$

This differential operator  $\nabla_{\rho,\tau}$  is called the Schmid operator. Then  $\nabla_{\rho,\tau}$  can be decomposed as  $\nabla_{\rho,\tau}^+ \oplus \nabla_{\rho,\tau}^-$  with  $\nabla_{\rho,\tau}^\pm : C_{\rho,\tau}^\infty(R_J \backslash G/K) \rightarrow C_{\rho,\tau \otimes \text{Ad}_{\mathfrak{p}_{\pm}}}^\infty(R_J \backslash G/K)$  corresponding to the decomposition  $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ . For each  $\beta \in \Delta_n^+$ , the shift operator  $\nabla_{\rho,\tau_\lambda}^{\pm\beta} : C_{\rho,\tau_\lambda}^\infty(R_J \backslash G/K) \rightarrow C_{\rho,\tau_{\lambda \pm \beta}}^\infty(R_J \backslash G/K)$  is defined as the composition of

$\nabla_{\rho, \tau_\lambda}^\pm$  with the projector  $P^{\pm\beta}$  from  $V_{\tau_\lambda} \otimes \mathfrak{p}_\pm$  into the irreducible component  $V_{\tau_\lambda \pm \beta}$ ;  $\nabla_{\rho, \tau_\lambda}^{\pm\beta} = (1_{\mathcal{F}_\rho} \otimes P^{\pm\beta}) \nabla_{\rho, \tau_\lambda}^\pm$ .

On the other hand, the Casimir element  $\Omega$  is defined by  $\Omega = \sum X_i - \sum Y_j$ , where  $\{Y_j\}$  is an orthonormal basis of  $\mathfrak{k}$  with respect to the Killing form of  $\mathfrak{g}$ . It is well known that  $\Omega$  is in the center  $Z(\mathfrak{g}_\mathbb{C})$  of the universal enveloping algebra of  $\mathfrak{g}_\mathbb{C}$ .

**4.2. Differential equations.** In this subsection, we consider the system of differential equations satisfied by the Fourier-Jacobi type spherical functions.

First we discuss the case of the  $P_J$ -principal series representation  $\pi \in \hat{G}$  and the corner  $K$ -type  $\tau^*$ . It is well known that the Casimir element  $\Omega \in Z(\mathfrak{g}_\mathbb{C})$  acts on  $\pi$ , hence on  $\mathcal{J}_{\rho, \pi}(\tau)$ , as the scalar operator  $\chi_\Omega$  (cf. Knapp[4; Corollary 8.14]). Let  $\pi = \text{Ind}_{P_J}^G(\sigma \otimes \nu_z \otimes 1_{N_J})$  with data  $\sigma = (\varepsilon, D_n^+)$ ,  $\varepsilon(\gamma) = (-1)^n$ , and  $\tau^* = \tau_\lambda^*$  be the corner  $K$ -type of  $\pi$ , i.e.  $\lambda = (-n, -n)$ . Since  $\tau_{\lambda+(2,2)}^* = \tau_{(n-2, n-2)} \in \hat{K}$  does not occur in the  $K$ -types of  $\pi$  from Lemma 2.4, an element in  $\mathcal{J}_{\rho, \pi}(\tau)$  is annihilated by the action of the composition of the shift operators

$$\nabla_{\rho, \tau_{\lambda+(2,0)}}^{(0,2)} \circ \nabla_{\rho, \tau_\lambda}^{(2,0)} : C_{\rho, \tau_\lambda}^\infty(R_J \backslash G/K) \rightarrow C_{\rho, \tau_{\lambda+(2,2)}}^\infty(R_J \backslash G/K).$$

Hence we have a system of differential equations satisfied by  $f$  in  $\mathcal{J}_{\rho, \pi}(\tau)$ ;

$$(4.1) \quad \begin{cases} \Omega f = \chi_\Omega f, \\ \nabla_{\rho, \tau_{\lambda+(2,0)}}^{(0,2)} \circ \nabla_{\rho, \tau_\lambda}^{(2,0)} f = 0. \end{cases}$$

Let  $\pi = \text{Ind}_{P_J}^G(\sigma \otimes \nu_z \otimes 1_{N_J})$  with data  $\sigma = (\varepsilon, D_n^+)$ ,  $\varepsilon(\gamma) = -(-1)^n$ , and  $\tau^* = \tau_\lambda^*$  be the corner  $K$ -type of  $\pi$ , i.e.  $\lambda = (-n+1, -n)$ . Since  $\tau_{\lambda+(1,1)}^* = \tau_{(n-2, n-1)} \in \hat{K}$  does not occur in the  $K$ -types of  $\pi$  from Lemma 2.4, therefore an element in  $\mathcal{J}_{\rho, \pi}(\tau)$  vanishes by the action of the shift operator

$$\nabla_{\rho, \tau_{\lambda+(1,1)}}^{(1,1)} : C_{\rho, \tau_\lambda}^\infty(R_J \backslash G/K) \rightarrow C_{\rho, \tau_{\lambda+(1,1)}}^\infty(R_J \backslash G/K).$$

Hence we have a system of differential equations satisfied by  $f$  in  $\mathcal{J}_{\rho, \pi}(\tau)$ ;

$$(4.2) \quad \begin{cases} \Omega f = \chi_\Omega f, \\ \nabla_{\rho, \tau_{\lambda+(1,1)}}^{(1,1)} f = 0. \end{cases}$$

For the case with the data  $\sigma = (\varepsilon, D_n^-)$ , we have similar systems of equations from the Casimir operator and the shift operators.

Let  $\pi = \pi_\Lambda$  be a discrete series representation of  $G$  with the Harish-Chandra parameter  $\Lambda \in \Xi_J$  and  $\tau^* = \tau_\lambda^* \in \hat{K}$  be the minimal  $K$ -type of  $\pi$ . Now we refer the following proposition which enables us to identify the intertwining space  $\mathcal{I}_{\rho, \pi}$  with a solution space of differential equations for any  $\rho \in \hat{R}_J$ .

**Proposition 4.1.** (Yamashita [9; Theorem 2.4]) *Let  $\pi = \pi_\Lambda \in \hat{G}$  and  $\tau^* = \tau_\lambda^* \in \hat{K}$  be as above. Then we have a linear isomorphism*

$$\mathcal{I}_{\rho, \pi} \simeq \bigcap_{\beta \in \Delta_{J^*, n}^+} \ker(\nabla_{\rho, \tau}^{-\beta}) \subset C_{\rho, \tau}^\infty(R_J \backslash G/K)$$



for any  $\rho \in \hat{R}_J$ . In particular,

$$\mathcal{J}_{\rho,\pi}(\tau) = \{F \in C_{\rho,\tau}^\infty(R_J \backslash G/K) \mid \nabla_{\rho,\tau}^{-\beta} F = 0, \quad \forall \beta \in \Delta_{J^*,n}^+\}.$$

Here the index  $J^*$  means IV, III, II and I for  $J = \text{I, II, III and IV}$ , respectively.

### 5. Result

Solving the systems of the differential equations given by (4.1), (4.2) and Proposition 4.1, we obtain the following theorem.

**Theorem 5.1.** *Let  $\pi$  be a  $P_J$ -principal series representation (resp. a discrete series representation) of  $G = Sp(2, \mathbf{R})$  and  $\tau^*$  be the 'corner'  $K$ -type (resp. the minimal  $K$ -type) of  $\pi$ . For each irreducible unitary representation  $\rho$  of  $R_J$  of type  $m \neq 0$ , we have*

$$\dim \mathcal{J}_{\rho,\pi}^\circ(\tau) \leq 1.$$

Moreover the radial parts of the functions in  $\mathcal{J}_{\rho,\pi}^\circ(\tau)$  are expressed by the Meijer's  $G$ -function  $G_{2,3}^{3,0} \left( x \left| \begin{smallmatrix} a_1, a_2 \\ b_1, b_2, b_3 \end{smallmatrix} \right. \right)$  or more degenerate similar functions.

Here the Meijer's  $G$ -function  $G_{2,3}^{3,0}(x) = G_{2,3}^{3,0} \left( x \left| \begin{smallmatrix} a_1, a_2 \\ b_1, b_2, b_3 \end{smallmatrix} \right. \right)$  with the complex parameters  $a_i, b_j$  ( $1 \leq i \leq 2, 1 \leq j \leq 3$ ) is the many-valued function defined by the integral

$$G_{2,3}^{3,0}(x) = G_{2,3}^{3,0} \left( x \left| \begin{smallmatrix} a_1, a_2 \\ b_1, b_2, b_3 \end{smallmatrix} \right. \right) = \frac{1}{2\pi\sqrt{-1}} \int_L \frac{\prod_{j=1}^3 \Gamma(b_j - t)}{\prod_{i=1}^2 \Gamma(a_i - t)} x^t dt$$

of Mellin-Barnes type, where the contour  $L$  is a loop starting and ending at  $+\infty$  and encircling all poles of  $\Gamma(b_j - t)$  ( $1 \leq j \leq 3$ ) once in the negative direction. It is known that, up to constant multiple,  $G_{2,3}^{3,0}(x)$  is the unique solution of the linear differential equation of 3-rd order

$$\left\{ x^3 \frac{d^3}{dx^3} + \alpha_2(x) x^2 \frac{d^2}{dx^2} + \alpha_1(x) x \frac{d}{dx} + \alpha_0(x) \right\} y = 0$$

with

$$\begin{aligned} \alpha_2(x) &= 3 - b_1 - b_2 - b_3 + x, \\ \alpha_1(x) &= (1 - b_1)(1 - b_2)(1 - b_3) + b_1 b_2 b_3 + (3 - a_1 - a_2)x, \\ \alpha_0(x) &= -b_1 b_2 b_3 + (1 - a_1)(1 - a_2)x, \end{aligned}$$

which decays exponentially as  $|x| \rightarrow \infty$  in  $-\frac{3}{2}\pi < \arg x < \frac{1}{2}\pi$  (See the Meijer's original paper [5] for details).

**Remark 5.2.** Let  $\pi$  be a holomorphic discrete series representation of  $G$  and  $\tau^*$  be the minimal  $K$ -type of  $\pi$ . Moreover, put  $\rho = \pi_1 \otimes \tilde{\nu}_m \in \hat{R}_J$  as in §2. For each  $m \neq 0$ , there is at most finitely many  $\rho$  such that  $\dim \mathcal{J}_{\rho,\pi}^\circ(\tau) = 1$ , and then the  $\pi_1$ -factors of such  $\rho$ 's are the holomorphic discrete series representations of  $\widetilde{SL}(2, \mathbf{R})$ . Moreover, the radial parts of the functions in  $\mathcal{J}_{\rho,\pi}^\circ(\tau)$  are expressed by the function of the form  $x^p e^{qx}$  for some constant  $p, q$ .

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